Exact Polynomial Reproduction for Oscillatory Radial Basis Functions on Infinite Lattices

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Abstract

Until now, only non-oscillatory radial basis functions (RBFs) have been considered in the literature. It has recently been shown that a certain family of oscillatory RBFs based on J - Bessel functions give rise to non singular interpolation problems and seem to be the only class of functions not to diverge in the limit of flat basis functions for any node layout. This paper proves another interesting feature of these functions: exact polynomial reproduction of arbitrary order on an infinite lattice in $\mathbb{R}^n$. First, a closed form expression is derived for calculating the expansion coefficients for any order polynomial in any dimension . Then, a proof is given showing that the resulting interpolant, using this class of oscillatory RBFs, will give exact polynomial reproduction. Examples in one and two dimensions are presented. It is specifically noted that such closed form expressions can not be derived for other classes of RBFs due to the fact that J - Bessel RBFs reproduce polynomials via a different mechanism.

1 Introduction

Given distinct nodes $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ in $\mathbb{R}^n$ and corresponding (scalar) function values $f_1, f_2, \ldots, f_n$, a radial basis function (RBF) interpolant to this data is given by

$$s(\bar{x}) = \sum_{j=1}^{n} \lambda_j \varphi(\|\bar{x} - \bar{x}_j\|_2) \quad (1)$$

where $r = \| \cdot \|_2$ is the standard Euclidean vector norm and $\varphi(r)$ is a radial function. The expansion coefficient λ_j are chosen to satisfy the interpolation conditions.

Until recently, the radial functions $\varphi(r)$ in (1) that have been considered in the literature have been non-oscillatory. However, it is noted in Fornberg et. al [5], that the family of radial functions

$$\varphi_d(\epsilon r) = \frac{J_{\frac{d}{2}-1}(\epsilon r)}{(\epsilon r)^{\frac{d}{2}-1}} \ , \quad d = 2, 3, 4, 5 \ldots \quad (2)$$

possess special properties. Here, $J_\nu(r)$ denotes the J-Bessel function of order ν and ϵ is a free 'shape' parameter. This class includes Gaussians in the special limit as $(d, \epsilon) \to \infty$, $\lim_{\delta\to\infty} 2^\delta \delta! \frac{J_\delta(2\sqrt{\delta}r)}{(2\sqrt{\delta}r)^\delta} = e^{-r^2}$, where $\delta = \frac{d}{2} - 1$ and $\epsilon = 2\sqrt{\delta}$. Similar to several other classes of RBFs, the resulting interpolation matrices (here, in up to d dimensions) are positive definite, assuring nonsingular interpolation. The class of Bessel RBFs, given by (2), is unusual in that

1. There is strong evidence to suggest they are the only type of RBFs to feature unconditional non-divergence of interpolants for any node distribution in $n \leq d$ dimensions, in the flat limit as $\epsilon \to 0$.

2. They are band limited in Fourier space.

The first property that is noted above is relevant in that, as $\epsilon \to 0$, RBF interpolation can reproduce all previous classical pseudospectral methods [3] [4], in some cases stably computable by the contour-Padé algorithm in [6]. Fornberg et. al [5] proved that interpolants using $\varphi_d(r)$ never diverge in the particularly severe test case of all data lying along a line, and the interpolant being evaluated off that line. Except for Gaussians, which is the limiting case of $\varphi_d(r)$ as noted above, they proved that no other classically used RBF in the literature possesses this property. For the Gaussian case, it was conjectured that no matter how points are scattered in any number of dimensions, Gaussian interpolants will never diverge in the limit as $\epsilon \to 0$. This was later proven in [10].

The second property, noted above, is unusual in that there are no currently used RBFs that are band limited in Fourier space. This property is the cornerstone for proving that Bessel RBF interpolants give polynomial reproduction on unbounded lattices in $\mathbb{R}^n$. Unlike commonly used RBFs, such as multiquadrics, the Bessel RBF interpolant can reproduce a polynomial of any order in a given dimension.

The remainder of the paper is organized as follows: In Section 2, the polynomial reproduction properties of some of the commonly used RBFs is reviewed; Section 3 derives the closed-form expression for the Bessel RBF expansion coefficients for the interpolant and gives a proof of exact polynomial reproduction in $\mathbb{R}^n$. Section 4 concludes the paper with examples in one and two dimensions.

Note: The explicit formulas for RBF expansion coefficients presented in this paper illustrate the uniqueness of Bessel RBFs. For other RBFs, no closed-form expressions for the expansion coefficients can be given in the cases where exact polynomial reproduction has been observed. In those cases, as the interval continually becomes wider, every individual coefficient associated with interior nodes will tend to zero . All non-trivial information will come from coefficients located towards the ends of the continually increasing interval. The mechanism by which Bessel RBFs reproduce polynomials is fundamentally different. For example, in the case of reproducing a constant (see Section 4), all the expansion coefficients are equal and non-zero.

2 Polynomial reproduction properties of commonly used RBFs

Powell [9] and Buhmann [2] summarize the polynomial reproduction properties of the commonly used RBFs. Powell proves, via his Theorems 5.2, 5.3, and 7.2, and Buhmann in Section 4.1 of his book that as the dimension increases so does the polynomial that can be recovered by the RBF interpolant except in the case of Gaussian RBFs. With Π_μ^n denoting the set of all polynomials of at most degree μ in n dimensions (e.g. $\Pi_2^2 = \{1, x, y, xy, x^2, y^2\}$), Table 2 summarizes the values of μ for which the RBF interpolant, $s(\bar{x})$, is such that $s(\bar{x}) \equiv f(\bar{x})$, $f \in \Pi_\mu^n$. The entries assume ϵ is a positive constant. For example, in one dimension, $n = 1$, inverse multiquadrics and quadratics can not reproduce polynomials of any order while multiquadrics can only reproduce a constant and linear function. Interestingly, in three dimensions a cubic can reproduce all polynomials up to quintic order and thin plate splines up to quadratic order. Unlike the above examples, we will prove in the following section that the family of Bessel RBFs, given by (2), can reproduce a polynomial of any order in a given dimension.

3 Exact polynomial reproduction in $\mathbb{R}^n$

As explained in the introduction, a key point of this paper is not simply to present nice exact expressions for the expansion coefficients with regard to polynomial reproduction using Bessel RBFs, but to specifically note that this class of RBFs is unique in that similar formulas *can not* be given for other classes of RBFs for which exact polynomial reproduction has been observed.

RBF	$\phi(r)$	μ
Piecewise Smooth		
Linear	r	n
Cubic	r^3	$n+1$
Thin Plate Spline	$r^2\log(r)$	$n+2$
Infinitely Smooth		
Multiquadrics	$\sqrt{1+(\varepsilon r)^2}$	n
Inverse Multiquadrics	$\frac{1}{\sqrt{1+(\varepsilon r)^2}}$	$n-2$
Inverse Quadratics	$\frac{1}{1+(\varepsilon r)^2}$	$n-3$
Gaussians	$e^{-\varepsilon r^2}$	...

Table 1: Summary of the degree, μ, of the polynomial that can be reproduced in a given dimension n for some common RBFs

Given this unique feature, the RBF Bessel expansion coefficients will first be derived in closed form for reproducing a polynomial of degree m in $\mathbb{R}^n$ on a regular infinite lattice of spacing h. This result will then be used in the following section to provide a proof that the resulting Bessel RBF expansion yields exact polynomial reproduction.

3.1 Bessel RBF Expansion Coefficients for Polynomial Interpolation

Given the vectors $\bar{x} = [x_1, x_2, ..., x_n]$ and $\bar{m} = [m_1, m_2, \ldots m_n]$, let us consider the polynomial $f(\bar{x}) = x_1^{m_1} x_2^{m_2} ... x_N^{m_n} = \bar{x}^{\bar{m}}$. If we let $\bar{x} = \bar{l}h$ on a discrete grid, $\bar{l} = [l_1, l_2, \ldots, l_n]$, of spacing h, then the Bessel RBF expansion of $\bar{x}^{\bar{m}}$ on this infinite n-dimensional lattice is

$$(\bar{l}h)^{\bar{m}} = \sum_{\bar{k}=-\infty}^{\infty} \lambda(\bar{k}) \varphi_d(\epsilon h ||\bar{l} - \bar{k}||_2) \quad \bar{l}, \bar{k} \in \mathbb{Z}^n \tag{3}$$

where $\varphi_d(\epsilon r) = \frac{J_{d/2-1}(\epsilon r)}{(\epsilon r)^{d/2-1}}$. The sum in (3) can be viewed as a discrete convolution which, in Fourier space, can be written as (cf. Arsac [1])

$$\sum_{\bar{l}=-\infty}^{\infty} (\bar{l}h)^{\bar{m}} e^{-ih(\bar{l}\cdot\bar{\xi})} = \left(\sum_{\bar{k}=-\infty}^{\infty} \lambda(\bar{k}) e^{-ih(\bar{k}\cdot\bar{\xi})} \right) \left(\sum_{\bar{p}=-\infty}^{\infty} \varphi_d(\epsilon h ||\bar{p}||_2) e^{-ih(\bar{p}\cdot\bar{\xi})} \right),$$

having set $\bar{p} = \bar{l} - \bar{k}$. Taking the inverse Fourier transform with respect to $\bar{\xi} = [\xi_1, \xi_2, ..., \xi_n]$ gives an integral expression for the coefficients $\lambda(\bar{k})$,

$$\lambda(\bar{k}) = \left(\frac{h}{2\pi}\right)^n \int_{[-\frac{\pi}{h},\frac{\pi}{h}]^n} \frac{\sum_{\bar{l}=-\infty}^{\infty} (\bar{l}h)^{\bar{m}} e^{-ih(\bar{l}\cdot\bar{\xi})}}{\sum_{\bar{p}=-\infty}^{\infty} \varphi_d(\epsilon h||\bar{p}||_2) e^{-ih(\bar{p}\cdot\bar{\xi})}} e^{ih(\bar{k}\cdot\bar{\xi})} d\bar{\xi}, \quad (4)$$

noting that the frequencies that can be represented on the defined lattice are $\bar{\xi} \in [-\frac{\pi}{h}, \frac{\pi}{h}]$. In order to find a closed form expression for (4), the sums need to be evaluated.

First let us examine the sum in the numerator of (4).

$$\sum_{\bar{l}=-\infty}^{\infty} (\bar{l}h)^{\bar{m}} e^{-ih(\bar{l}\cdot\bar{\xi})} = \sum_{\bar{l}=-\infty}^{\infty} \left[(l_1 h)^{m_1} e^{-ihl_1\xi_1}\right] \left[(l_2 h)^{m_2} e^{-ihl_2\xi_2}\right] ... \left[(l_n h)^{m_n} e^{-ihl_n\xi_n}\right]. \quad (5)$$

Each expression in the square brackets in the sum (5) can be re-written. For example in the first set of brackets,

$$(l_1 h)^{m_1} e^{-ihl_1\xi_1} = \frac{1}{(-i)^{m_1}} \frac{\partial^{m_1}}{\partial \xi_1^{m_1}} \left(e^{-ihl_1\xi_1}\right).$$

This results in

$$\sum_{\bar{l}=-\infty}^{\infty} (\bar{l}h)^{\bar{m}} e^{-ih(\bar{l}\cdot\bar{\xi})} = \frac{1}{(-i)^{m_1}...(-i)^{m_n}} \sum_{\bar{l}=-\infty}^{\infty} \frac{\partial^{m_1}...\partial^{m_n}}{\partial\xi_1^{m_1}...\partial\xi_n^{m_n}} e^{-ih(\bar{l}\cdot\bar{\xi})} \quad (6)$$

$$= \frac{1}{(-i)^{m_1}...(-i)^{m_n}} \frac{\partial^{m_1}...\partial^{m_n}}{\partial\xi_1^{m_1}...\partial\xi_n^{m_n}} \sum_{\bar{l}=-\infty}^{\infty} e^{-ih(\bar{l}\cdot\bar{\xi})}. \quad (7)$$

The Poisson summation formula (cf. Lighthill [8]) allows one to re-write the discrete Fourier transform (FT) of a function as its continuous FT summed over a $\frac{2\pi}{h}$ periodic grid. The formula, when applied to (7), results in a multidimensional $\delta-$function that repeats itself with a spacing of $\frac{2\pi}{h}$ over the $\mathbb{Z}^n$ lattice. That is,

$$\sum_{\bar{l}=-\infty}^{\infty} e^{-ih(\bar{l}\cdot\bar{\xi})} = \left(\frac{2\pi}{h}\right)^n \sum_{\bar{q}=-\infty}^{\infty} \delta\left(\bar{\xi} - \frac{2\pi\bar{q}}{h}\right).$$

However, since $\bar{\xi}$ is confined to $[-\frac{\pi}{h}, \frac{\pi}{h}]$ in all dimensions, only the term $\bar{q} = \bar{0}$ will make an contribution. Thus,

$$\sum_{\bar{l}=-\infty}^{\infty} (\bar{l}h)^{\bar{m}} e^{-ih(\bar{l}\cdot\bar{\xi})} = \left(\frac{2\pi}{h}\right)^n \frac{1}{(-i)^{m_1}...(-i)^{m_n}} \frac{\partial^{m_1}\partial^{m_2}...\partial^{m_n}}{\partial\xi_1^{m_1}\partial\xi_2^{m_2}...\partial\xi_n^{m_n}} \delta\left(\bar{\xi}\right). \quad (8)$$

In order to simplify the notation, the following definitions are made

$$\frac{\partial^{\bar{m}}}{\partial \bar{\xi}^{\bar{m}}} = \frac{\partial^{m_1}\partial^{m_2}...\partial^{m_n}}{\partial \xi_1^{m_1}\partial \xi_2^{m_2}...\partial \xi_n^{m_n}}$$

$$\frac{1}{(-i)^{\bar{m}}} = \frac{1}{(-i)^{m_1}(-i)^{m_2}...(-i)^{m_n}}.$$

Substituting (8) into (4), gives

$$\lambda(\bar{k}) = \frac{1}{(-i)^{\bar{m}}} \int\limits_{[-\frac{\pi}{h},\frac{\pi}{h}]^n} \left\{ \frac{\partial^{\bar{m}}}{\partial \bar{\xi}^{\bar{m}}} \delta\left(\bar{\xi}\right) \right\} \frac{e^{ih(\bar{k}\cdot\bar{\xi})}}{\sum_{\bar{p}=-\infty}^{\infty} \varphi_d(\epsilon h||\bar{p}||_2) e^{-ih(\bar{p}\cdot\bar{\xi})}} d\bar{\xi} \quad (9)$$

$$= \frac{1}{i^{\bar{m}}} \frac{\partial^{\bar{m}}}{\partial \bar{\xi}^{\bar{m}}} \left\{ \frac{e^{ih(\bar{k}\cdot\bar{\xi})}}{\sum_{\bar{p}=-\infty}^{\infty} \varphi_d(\epsilon h||\bar{p}||_2) e^{-ih(\bar{p}\cdot\bar{\xi})}} \right\}\Bigg|_{\bar{\xi}=\bar{0}} \quad (10)$$

where the following property of $\delta-$functions has been used (cf. Lighthill [8])

$$\int_{-\infty}^{\infty} \delta^{(m)}(\omega)F(\omega)d\omega = (-1)^m F^{(m)}(0).$$

Next, the denominator in (10) needs to be evaluated, first using the Poisson summation formula

$$\sum_{\bar{p}=-\infty}^{\infty} \varphi_d(\epsilon h||\bar{p}||_2)e^{-ih(\bar{p}\cdot\bar{\xi})} = \left(\frac{2\pi}{h}\right)^n \sum_{\bar{q}=-\infty}^{\infty} \hat{\varphi}_d\left(\left\|\frac{\bar{\xi}}{\epsilon} - \frac{2\pi\bar{q}}{\epsilon h}\right\|_2\right) \quad (11)$$

where $\hat{\varphi}_d$ is the continuous FT of φ_d in $\mathbb{R}^n$, defined as

$$\hat{\varphi}_d\left(\frac{||\bar{\xi}||_2}{\epsilon}\right) = \int\limits_{\mathbb{R}^n} \varphi_d(\epsilon||\bar{x}||_2)e^{-i(\bar{x}\cdot\bar{\xi})}d\bar{x} \quad (12)$$

$$= \frac{\left(1-\frac{||\bar{\xi}||_2^2}{\epsilon^2}\right)^{\left(\frac{d-n}{2}-1\right)}}{\epsilon^n 2^{(d/2-1)}\Gamma\left(\frac{d-n}{2}\right)\pi^{n/2}}, \quad -1 < \frac{\xi_1}{\epsilon}, \dots, \frac{\xi_n}{\epsilon} < 1 \quad (13)$$

$$= 0 \quad \left|\frac{\xi_1}{\epsilon}\right|, \dots, \left|\frac{\xi_n}{\epsilon}\right| \geq 1 \quad (14)$$

Equations (13) - (14) show that the Bessel RBFs are band limited (i.e. compact in Fourier space). Thus, if $\hat{\varphi}_d$ is to be non-zero in (11), then for $\epsilon h \leq 2\pi$ the only term that enters into the summation is $\bar{q} = \bar{0}$. As a result, the sum in (11) collapses to

$$\sum_{\bar{p}=-\infty}^{\infty} \varphi_d(\epsilon h||\bar{p}||_2)e^{-ih(\bar{p}\cdot\bar{\xi})} = \frac{\pi^{n/2}\left(1-\frac{\bar{\xi}^2}{\epsilon^2}\right)^{\left(\frac{d-n}{2}-1\right)}}{2^{(d/2-1)-n}\Gamma\left(\frac{d-n}{2}\right)(\epsilon h)^n}$$

giving the formula for the coefficients

$$\lambda(\bar{k}) = \frac{\left(2^{(d/2-1)-n}\right)\Gamma\left(\frac{d-n}{2}\right)(\epsilon h)^n}{i^{\bar{m}}\pi^{n/2}} \frac{\partial^{\bar{m}}}{\partial\bar{\xi}^{\bar{m}}} \left\{ \frac{e^{ih(\bar{k}\cdot\bar{\xi})}}{\left(1-\frac{\bar{\xi}^2}{\epsilon^2}\right)^{(d/2-1)-n/2}} \right\}\Bigg|_{\bar{\xi}=\bar{0}} \tag{15}$$

$$= \frac{1}{i^{\bar{m}}}\left(\frac{h}{2\pi}\right)^n \frac{\partial^{\bar{m}}}{\partial\bar{\xi}^{\bar{m}}} \left\{ \frac{e^{ih(\bar{k}\cdot\bar{\xi})}}{\hat{\varphi}_d\left(\frac{||\bar{\xi}||_2}{\epsilon}\right)} \right\}\Bigg|_{\bar{\xi}=\bar{0}}. \tag{16}$$

It needs to be noted that when using $\varphi_d(r)$ in n dimensions to reproduce $\bar{x}^{\bar{m}}$, the individual terms in the sum, $s(\bar{x}) = \sum_{\bar{k}=-\infty}^{\infty} \lambda(\bar{k})\varphi_d(\epsilon||\bar{x}-h\bar{k}||_2)$, go to zero if $d > 2m+1$ and the sum converges absolutely if $d > 2(m+n)+1$, where the scalar $m = m_1 + m_2 + \ldots + m_n$. Since the radial functions are oscillatory (and furthermore, the expansion coefficients exhibit cancellations if any of the powers m_i are odd), absolute convergence is not necessary for conditional convergence. We will see an example of this in the 2D case in Section 4.

3.2 Proof of Exact Reproduction of $\bar{x}^{\bar{m}}$ for Bessel RBF Expansions on Infinite Grids

Theorem 1 *Given* $f(\bar{x}) = x_1^{m_1}x_2^{m_2}...x_N^{m_n} = \bar{x}^{\bar{m}}$, $\bar{m} \in \mathbb{N}^n$ *(including 0) and* $\lambda(\bar{k})$ *from (15), then the Bessel RBF interpolant,*

$$s(\bar{x}) = \frac{\left(2^{(d/2-1)-n}\right)\Gamma\left(\frac{d-n}{2}\right)(\epsilon h)^n}{i^{\bar{m}}\pi^{n/2}} \sum_{\bar{k}=-\infty}^{\infty} \frac{\partial^{\bar{m}}}{\partial\bar{\xi}^{\bar{m}}} \left\{ \frac{e^{ih(\bar{k}\cdot\bar{\xi})}}{\left(1-\frac{\bar{\xi}^2}{\epsilon^2}\right)^{(d/2-1)-n/2}} \right\}\Bigg|_{\bar{\xi}=\bar{0}} \varphi_d(\epsilon||\bar{x}-h\bar{k}||_2),$$

to this function on an infinite grid of spacing h *is exact for* $\epsilon h \leq 2\pi$*, when* d *is sufficiently large that the sum converges.*

Proof. For easier reading we will use the representation of the expansion coefficients given by (16), and then interchange the order of summation and differentiation. The expression for the interpolant becomes

$$s(\bar{x}) = \frac{1}{i^{\bar{m}}}\left(\frac{h}{2\pi}\right)^n \sum_{\bar{k}=-\infty}^{\infty} \left(\frac{\partial^{\bar{m}}}{\partial\bar{\xi}^{\bar{m}}} \left\{ \frac{e^{ih(\bar{k}\cdot\bar{\xi})}}{\hat{\varphi}_d\left(\frac{||\bar{\xi}||_2}{\epsilon}\right)} \right\}\Bigg|_{\bar{\xi}=\bar{0}} \right) \varphi_d(\epsilon||\bar{x}-h\bar{k}||_2) \tag{17}$$

$$= \frac{1}{i^{\bar{m}}}\left(\frac{h}{2\pi}\right)^n \frac{\partial^{\bar{m}}}{\partial\bar{\xi}^{\bar{m}}} \left\{ \frac{1}{\hat{\varphi}_d\left(\frac{||\bar{\xi}||_2}{\epsilon}\right)} \sum_{\bar{k}=-\infty}^{\infty} e^{ih(\bar{k}\cdot\bar{\xi})}\varphi_d(\epsilon||\bar{x}-h\bar{k}||_2) \right\}\Bigg|_{\bar{\xi}=\bar{0}} \tag{18}$$

Notice that $\sum_{\bar{k}=-\infty}^{\infty} e^{ih(\bar{k}\cdot\bar{\xi})}\varphi_d(\bar{x}-h\bar{k})$ is the discrete convolution of two functions. So we can again use Poisson's summation formula along with the commutative law for convolutions to rewrite the summation as

$$\begin{aligned}\sum_{\bar{k}=-\infty}^{\infty} e^{ih(\bar{k}\cdot\bar{\xi})}\varphi_d(\epsilon||\bar{x}-h\bar{k}||_2) &= \sum_{\bar{k}=-\infty}^{\infty} e^{i\left(\bar{\xi}\cdot(\bar{x}-h\bar{k})\right)}\varphi_d(\epsilon h\left|\left|\bar{k}\right|\right|_2) \\ &= \left(\frac{2\pi}{h}\right)^n e^{i(\bar{\xi}\cdot\bar{x})}\sum_{\bar{j}=-\infty}^{\infty}\hat{\varphi}_d\left(\left\|\frac{\bar{\xi}}{\epsilon}-\frac{2\pi\bar{j}}{\epsilon h}\right\|_2\right).\end{aligned}$$

Recalling that the FT of the Bessel RBF is compact on $(-1,1)$ and that $\bar{\xi}\in\left[-\frac{\pi}{h},\frac{\pi}{h}\right]$, then only the $\bar{j}=\bar{0}$ term will make a non-zero contribution to the summation for $\epsilon h\leq 2\pi$. As a note, if $\epsilon h>2\pi$, additional terms enter into the summation resulting in non-exact reproduction and oscillatory behavior of the error as will be numerically demonstrated in the examples. The interpolant, (18), now reduces to

$$\begin{aligned}s(\bar{x}) &= \frac{1}{i^{\bar{m}}}\left(\frac{h}{2\pi}\right)^n\frac{\partial^{\bar{m}}}{\partial\bar{\xi}^{\bar{m}}}\left\{\frac{1}{\hat{\varphi}_d\left(\frac{||\bar{\xi}||_2}{\epsilon}\right)}\left[\left(\frac{2\pi}{h}\right)^n e^{i(\bar{\xi}\cdot\bar{x})}\hat{\varphi}_d\left(\frac{||\bar{\xi}||_2}{\epsilon}\right)\right]\right\}\Bigg|_{\bar{\xi}=\bar{0}} \\ &= \frac{1}{i^{\bar{m}}}\frac{\partial^{\bar{m}}}{\partial\bar{\xi}^{\bar{m}}}e^{i(\bar{\xi}\cdot\bar{x})}\Bigg|_{\bar{\xi}=\bar{0}} \\ &= \bar{x}^{\bar{m}}.\end{aligned}$$

■

4 Examples

4.1 1-D case

In one dimension, the Bessel RBF expansion for $f(x)=x^m$ can be simply written as

$$s(x)=\frac{2^{(d/2-2)}\Gamma\left(\frac{d-1}{2}\right)\epsilon h}{\sqrt{\pi}i^m}\sum_{k=-\infty}^{\infty}\frac{\partial^m}{\partial\xi^m}\left\{\frac{e^{ih\xi k}}{\left(1-\frac{\xi^2}{\epsilon^2}\right)^{(d/2-3/2)}}\right\}\Bigg|_{\xi=0}\varphi_d(\epsilon|x-hk|)\cdot \tag{19}$$

Notice that the expansion coefficients $\lambda(k)$ will be a polynomial in k of degree m. Therefore, in order for the RBF expansion to be convergent, which was assumed in the proof above, we need to choose a d in $\varphi_d(\epsilon|x-hk|)=\frac{J_{d/2-1}(\epsilon|x-hk|)}{(\epsilon|x-hk|)^{d/2-1}}$ such that it decays sufficiently more rapidly than k^m. For the 1-D case, this can be done by letting $d>2m+1$. Thus, as the power of the polynomial increases so must the order of the Bessel RBF. The larger d is the faster the series converges.

Below, we give the coefficients for $f(x)=1,x,x^2,x^3$.

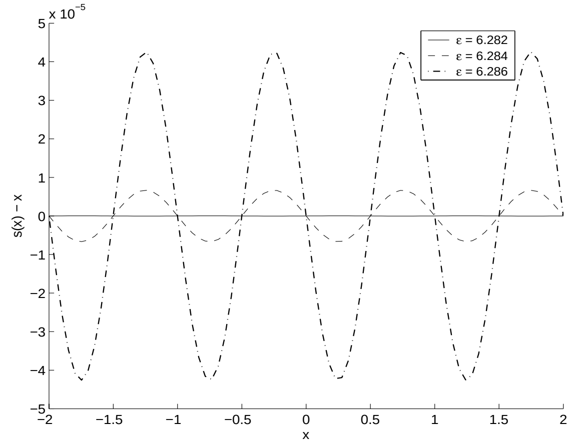

Figure 1: Error plot for the interpolant $s(x)$ to $f(x) = x$ when $\epsilon h < 2\pi$ and $\epsilon h > 2\pi$

$f(x)$	$\lambda(k)$
1	α
x	$\alpha h k$
x^2	$\alpha\left(h^2k^2 - \frac{d-3}{\epsilon^2}\right)$
x^3	$\alpha\left(h^3k^3 - \frac{3d-9}{\epsilon^2}hk\right)$

where the constant $\alpha = \frac{2^{(d/2-2)}\Gamma\left(\frac{d-1}{2}\right)\epsilon h}{\sqrt{\pi}}$. Notice that even polynomials will have expansion coefficients that are even polynomials in k and odd polynomials will have expansion coefficients that are odd polynomials in k.

Since the error plots look similar for any polynomial, we will only show the cases $f(x) = x$ and $f(x) = x^2$ as given in Figures 1 and 2. We have let $d = 8$ and $h = 1$. As stated in the proof of the theorem, $\epsilon h \leq 2\pi$ is required for the the RBF expansion to give exact polynomial reproduction. For the value of $\epsilon = 6.282$ we see that we do get exact reproduction of the polynomial however as soon as ϵh is even slightly greater than 2π exact reproduction is lost and the interpolant becomes oscillatory. This is due to the fact that the Poisson sum for the FT of $\varphi_d(\epsilon r)$ no longer collapses down to one term, introducing additional terms into the RBF expansion.

Bessel RBF expansions will diverge if d is not appropriately chosen as mentioned above. If we allow $d \leq 2m + 1$, the interpolant diverges as is shown in Figure 3, where we again interpolate $f(x) = x^2$ but this time with $d = 4$, $h = 1$, $\epsilon = 6.282$ or $s(x) = \sum_{k=-\infty}^{\infty} \lambda(k)\varphi_4(\epsilon|x - k|)$. For the $d = 4$ case, the coefficients are growing as $O(k^2)$ but the Bessel RBF is only decaying as $O(k^{-3/2})$, resulting in the displayed divergence.

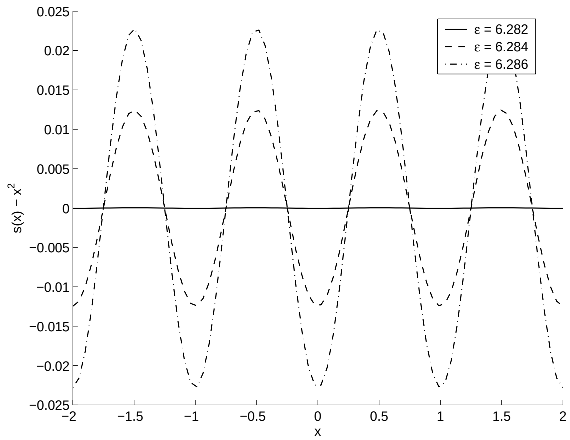

Figure 2: Error plot for the interpolant $s(x)$ to $f(x) = x^2$ when $\epsilon h < 2\pi$ and $\epsilon h > 2\pi$

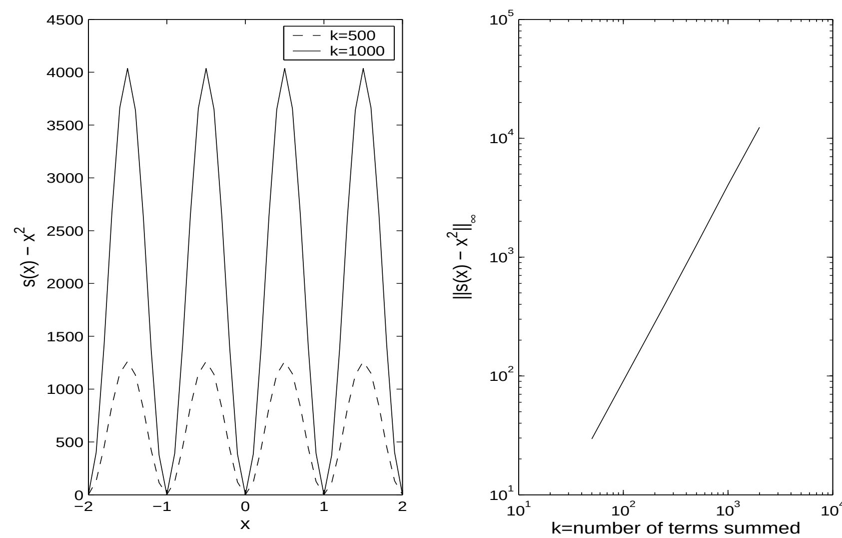

Figure 3: (a) The error in the Bessel RBF interpolant for x^2 for the case $d = 4$, $\epsilon = 6.282$, $h = 1$ when 500 (dashed line) and 1000 (solid line) terms are summed in (19). (b) L_∞ norm of the error as a function of the number of terms summed

4.2 2-D case

The 2-D case follows the same patterns observed in the 1-D case, except that the expansion coefficients, λ, are now a function of two variables $\bar{k} = [k_1, k_2]$. The interpolant is given by

$$s(x,y) = \beta \sum_{k_2=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} \frac{\partial^{(m_1+m_2)}}{\partial \xi_1^{m_1} \partial \xi_2^{m_2}} \left\{ \frac{e^{ih(\xi_1 k_1 + \xi_2 k_2)}}{\left(1 - \frac{\xi_1^2}{\epsilon^2} - \frac{\xi_2^2}{\epsilon^2}\right)^{(d/2-2)}} \right\}\Bigg|_{(\xi_1,\xi_2)=0} \varphi_d\left(\epsilon\sqrt{(x-hk_1)^2 + (x-hk_2)^2}\right), \tag{20}$$

where $\beta = \frac{2^{(d/2-3)}\Gamma\left(\frac{d-2}{2}\right)(\epsilon h)^2}{\pi i^{(m_1+m_2)}}$. In 2-D, of the infinitely smooth RBFs reviewed in Section 2, multiquadrics can reproduce the family of polynomials $\{1, x, y, xy, x^2, y^2\}$ and inverse multiquadrics can reproduce a constant. Again, Bessel RBFs can reproduce any polynomial $x^{m_1}y^{m_2}$, $m_1, m_2 \in \mathbb{N}$, so long as d is appropriately chosen and $\epsilon h \leq 2\pi$. For the 2-D case the expansion coefficients will grow as $O(k_1^{m_1}k_2^{m_2})$ as illustrated in the table below for the set of polynomials $\Pi_2^2 = \{1, x, y, xy, x^2, y^2\}$.

$f(x,y)$	$\lambda(k_1,k_2)$
1	β
x, y, xy	βhk_1, βhk_2, $\beta h^2 k_1 k_2$
x^2, y^2	$\beta\left(h^2k_1^2 - \frac{4-d}{\epsilon^2}\right)$, $\beta\left(h^2k_2^2 - \frac{4-d}{\epsilon^2}\right)$

However, as a more interesting example, we will consider the function $f(x,y) = x^3y^5$. In Figure 4, the error is plotted for $\epsilon h < 2\pi$ and ϵh just slightly over 2π with $d = 20$, $h = 1$. As in the 1-D case, exact reproduction properties are lost for values of $\epsilon h > 2\pi$. This is also a case where the expansion coefficients k_1 and k_2 alternate in sign due to being odd powers to highest order. This results in cancellation of terms that accelerates the convergence of the sum, (20), allowing for the use of a lower value of d than would be required for absolute convergence (in this case $\dot{d} > 21$).

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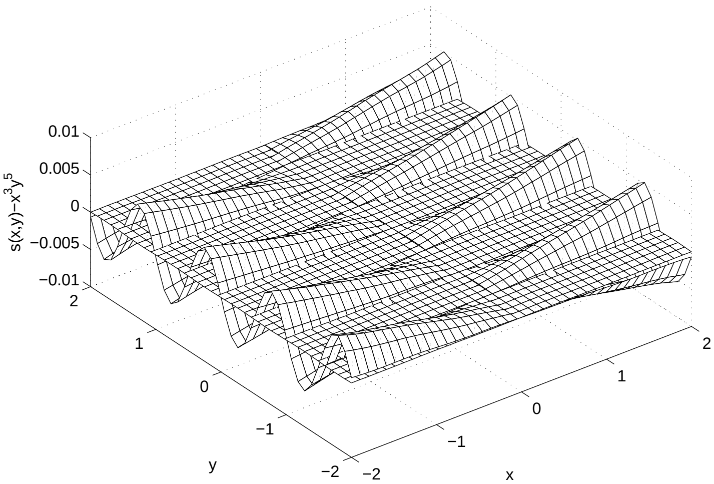

Figure 4: Error in the Bessel RBF interpolant to $f(x,y) = x^3y^5$. Flat surface is for $\epsilon = 6.28$. Oscillatory surface is for $\epsilon = 6.32$

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